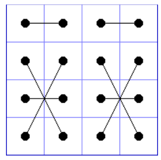
Connection figure (1122 3456 7788 4365)



Consider the following group of 8 permutations
P1,P2,...,P8 of {1,...,16} defined by

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
P1(x)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	identity
P2(x)	2	1	4	3	14	13	16	15	10	9	12	11	6	5	8	7	complement
P3(x)	4	3	2	1	8	7	6	5	12	14	10	9	16	15	14	13	vertical axis flip
P4(x)	3	4	1	2	15	16	13	14	11	12	9	10	7	8	5	6	
P5(x)	12	11	10	9	8	7	6	5	4	3	2	1	16	15	14	13	
P6(x)	11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6	
P7(x)	9	10	11	12	5	6	7	8	1	2	3	4	13	14	15	16	exchange of rows 1 and 3
P8(x)	10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7.	

Theorem 3.1

Let M = (ci), i=1,...,16, be a general 4x4 magic square with entries from a symmetric subset of {1,...,N} and connection figure (1122 3456 7788 4365), containing 1 as an entry.
Then there exists a triple (k,r,s) of natural numbers, such that M can be mapped by a permutation P1,P2,..., or P8, say Pj, onto a general 4x4 magic square M*=(cPj(i)), with the same connection figure (1122 3456 7788 4365), where either

(i) M* =

1	k+2r+3s	k	1+2r+3s
k+r+3s	1+r+2s	1+r+s	k+r
1+2r+2s	k+s	k+2r+2s	1+s
k+r+s	1+r	1+r+3s	k+r+2s

, or

(ii) M* =

1-r+2s	k+s	k-2r+2s	1+r+s
k-r+3s	1+2s	1+s	k-r
1+r	k-2r+3s	k	1-r+3s
k-r+s	1	1+3s	k-r+2s

.

Proof

Using only linear algebra, one can show, that every general 4x4 magic square with connection figure (1122 3456 7788 4365) may be written as

h	k+2r+3s	k	h+2r+3s
k+r+3s	h+r+2s	h+r+s	k+r
h+2r+2s	k+s	k+2r+2s	h+s
k+r+s	h+r	h+r+3s	k+r+2s

and as

h-r+2s	k+s	k-2r+2s	h+r+s
k-r+3s	h+2s	h+s	k-r
h+r	k-2r+3s	k	h-r+3s.
k-r+s	h	h+3s	k-r+2s

If the entry 1 belongs to the first or to the third row of M, then there exists a permutation from P1,P2,...,P8 which maps M onto a square of form M*, case (i).
If 1 is in the second or fourth row of M, then there are two permutations from P1,P2,...,P8, which bring M into the form M*, case (ii).

Theorem 3.2

(i) Let (k,r,s) be a triple of natural numbers (with r<>s), such that the 16 numbers

(*) 1,1+r,1+s,1+r+s,1+r+2s,1+r+3s,1+2r+2s,1+2r+3s,
k,k+r,k+s,k+r+s,k+r+2s,k+r+3s,k+2r+2s,k+2r+3s

are pairwise different.
Then (*) is a symmetric subset of {1,...,N}, N=k+2r+3s, and there are 8 different general 4x4 magic squares M01,M02,...,M08 with entries from (*) and connection figure (1122 3456 7788 8365), namely:

M01 =

1	k+2r+3s	k	1+2r+3s
k+r+3s	1+r+2s	1+r+s	k+r
1+2r+2s	k+s	k+2r+2s	1+s
k+r+s	1+r	1+r+3s	k+r+2s

, M02 = M01 +

s	-s	s	-s
-3s	-s	s	3s
s	-s	s	-s ,
s	3s	-3s	-s

M03 and M03 are derived from M01 and M02 by exchange of the symbols 1 and k, and M05, M06, M07 and M08 are the mirror images of M01,...,M04 by reflection at a vertical axis.

(ii) Let (k,r,s) be a triple of natural numbers (with $s < r < 2s$, r <> 1.5s), such that the 16 numbers

(**) 1,1+r,1+s,1+2s,1+r+s,1-r+2s,1-r+3s,1+3s,
k,k-r,k+s,k-r+s,k-r+2s,k-r+3s,k-2r+2s,k-2r+3s

are pairwise different.
Then (**) is a symmetric subset of {1,...,N}, N=k-r+3s, and there are 8 different general 4x4 magic squares M09,M10,...,M16 with entries from (**) and connection figure (1122 3456 7788 4365), namely:

$$M09 = \begin{matrix} 1-r+2s & k+s & k-2r+2s & 1+r+s \\ k-r+3s & 1+2s & 1+s & k-r \\ 1+r & k-2r+3s & k & 1-r+3s \\ k-r+s & 1 & 1+3s & k-r+2s, \end{matrix}$$

$$M10 = \begin{matrix} k-2r+2s & 1+r+s & 1-r+2s & k+s \\ 1+3s & k-r+2s & k-r+s & 1 \\ k & 1-r+3s & 1+r & k-2r+3s \\ 1+s & k-r & k-r+3s & 1+2s \end{matrix}$$

(in M09 the symbols k and 1+r were exchanged)

$$M11 = M09 + \begin{matrix} s & -s & s & -s \\ -3s & -s & s & 3s \\ s & -s & s & -s, \\ s & 3s & -3s & -s \end{matrix} \qquad M12 = M10 + \begin{matrix} s & -s & s & -s \\ -3s & -s & s & 3s \\ s & -s & s & -s, \\ s & 3s & -3s & -s \end{matrix}$$

and M13, M14, M15 and M16 are the mirror images of M01,...,M04 by reflection at a vertical axis.

<u>Proof</u>
This follows from Theorem 1, using the 8 permutations P1,P2,...,P8.

<u>Theorem 3.3</u>

Let T be a symmetric subset with of {1,...,N}, N<61, with 16 elements, containing the number 1, and let z1 be the number of triples (k,r,s) such that T consists of the 16 different numbers (*) from Theorem 2.
Moreover, let z2 the number of triples (k,r,s) with the property, that T consists of the different 16 numbers (**) from Theorem 2.
Then z1+z2 has the value either 0,1,2, or 4 and there are exactly 8z1 + 8z2 general 4x4 magic squares with connection figure (1122 3456 7788 4365) and entries from T.
Possible values for (z1,z2) are (0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(2,1),(1,2), (3,1) and (2,2).

<u>Proof</u> by computer experiment.

<u>Example</u>

N=18, k=2, r=5, s=2
Set (*): 1,6,3,8,10,12,15,17,
2,7,4,9,11,13,16,18

N=18, k=10, r=1, s=2
Set (*): 1, 2, 3, 4, 6, 8, 7, 9
10,11,12,13,15,17,16,18

N=18, k=12, r=9, s=5
Set (**): 1,10, 6,11,15, 2,7,16,
12, 3,17, 8,13,18,4, 9

For the set T={1,2,3,4,6,7,8,9,10,11,12,13,15,16,17,18} the values of z1, z2 are z1=2, z2=1, therefore T allows 24 general 4x4 magic squares of connection figure (1122 3456 7788 4365).

Remark

There is an injection i from the set of general 4x4 magic squares of connection figure (1122 3456 7788 4365) into the set of general magic squares with connection figure (1122 3344 5566 7788), defined as

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       c01 c02 c03 c04       c01 c02 c07 c16
       c05 c06 c07 c08       c09 c10 c15 c08
   i:  c09 c10 c11 c12  -->  c05 c14 c11 c12.
       c13 c14 c15 c16       c13 c06 c03 c04
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